Porous sets and Martin Numbers

Arturo Martínez-Celis

Centro de Ciencias Matemáticas, UNAM Hejnice, 2014

Arturo Martínez-Celis

UNAM

Let \mathbb{P} be a partial order. We say that \mathbb{P} is σ -centered if there is a family $\{P_i : i \in \omega\}$ of subsets of \mathbb{P} such that for every k and every $p, q \in P_k$ there is $p' \in P_k$ such that $p' \leq p$ and $p \leq q$.

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Definition

Let \mathbb{P} be a partial order. We say that \mathbb{P} is σ -*linked* if there is a family $\{P_i : i \in \omega\}$ of subsets of \mathbb{P} such that for every k and every $p, q \in P_k$ there is $p' \in \mathbb{P}$ such that $p' \leq p$ and $p \leq q$.

Let \mathbb{P} be a partial order. We say that \mathbb{P} is σ -*n*-linked if there is a family $\{P_i : i \in \omega\}$ of subsets of \mathbb{P} such that for every *k* and every sequence $\{p_i : i < n\} \subseteq P_k$ there is $p \in \mathbb{P}$ such that $p \leq p_i$ for every i < n.

Given a cardinal κ and a property φ about partial orders, we define $MA_{\varphi}(\kappa)$ as the following statement: For every partial order such that $\varphi(\mathbb{P})$ and every family of dense sets \mathcal{D} of \mathbb{P} such that $|\mathcal{D}| \leq \kappa$, there is a filter $F \subseteq \mathbb{P}$ such that F intersects every element of \mathcal{D} .

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$$\mathfrak{m}_{\sigma\text{-centered}} = \mathfrak{t}$$

 $\mathfrak{m}_{\textit{c.c.c.}} \leq \mathfrak{m}_{\sigma\text{-linked}} \leq \mathfrak{m}_{\sigma\text{-}3\text{-linked}} \leq \cdots \leq \mathfrak{m}_{\sigma\text{-centered}}$

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I'll try to construct a model where an infinite amount of these cardinal invariants are diferent.

A subset $A \subseteq 2^{\omega}$ is porous of degree *n* if for every $s \in 2^{<\omega}$ there is $t \in 2^n$ such that $\langle s \frown t \rangle \cap A = \emptyset$. A subset $A \subseteq 2^{\omega}$ is porous if it is porous of degree *n* for some $n \in \omega$.

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There are canonical porous sets: for every $\sigma: 2^{<\omega} \rightarrow 2^n$ let

$$X_{\sigma} = \{ x \in 2^{\omega} : orall k \in \omega(x \notin \langle x | k \frown \sigma(x | k)
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As a consequence we can use functions to code porous sets.

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It is easy to see that $Ctbl = SP_1 \subsetneq SP_2 \subsetneq SP_3 \subsetneq \cdots \subseteq SP$. Therefore $\omega_1 \le non(SP_2) \le non(SP_3) \le \cdots \le non(SP)$.

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Definition

We say that a partial order \mathbb{P} strongly preserves non(**SP**_{*n*}) if for every name X for a porous set of degree *n* such that \Vdash_P " $X \subseteq \check{2}^{\omega}$ ", there is $Y \in \mathbf{SP}_n$ such that \Vdash_P " $X \subseteq Y$ "

Lemma

Finite support iteration of c.c.c. forcings that strongly preserves $non(SP_n)$ strongly preserves $non(SP_n)$.

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Lemma

If \mathbb{P} is a σ -2^{*n*}-linked forcing, then \mathbb{P} strongly preserves non(**SP**_{*n*}).

$$\mathbb{P}_{n}^{0}(X) = \{ \langle s, F \rangle : \text{ (a) } s \text{ is a finite partial function from } 2^{<\omega} \text{ to } 2^{n}, \\ \text{ (b) } F \in [X]^{<\omega}, \\ \text{ (c) for each } \sigma \in \text{dom}(s), F \cap \langle \sigma \cap s(\sigma) \rangle = \emptyset, \\ \text{ (d) } F \text{ is a strongly porous set of degree } n \}$$

and define $\langle s, F \rangle \leq \langle s', F' \rangle$ if $s' \subseteq s$ and $F' \subseteq F$.

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Proposition

Let $\mathbb{P}_n(X) = (\mathbb{P}_n^0(X))^{<\omega}$. Then $\Vdash_{\mathbb{P}} ``\check{X} \in \mathbf{SP}_n"$.

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 $\mathbb{P}_n(X)$ is a σ -2^{*n*} – 1-linked forcing. As a consequence $\mathbb{P}_n(X)$ strongly preserves non(**SP**_{*n*}).

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Theorem

 $\mathfrak{m}_{\sigma-2^n-1}$ -linked $\leq \operatorname{non}(\mathbf{SP}_n)$

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Theorem

 $\mathfrak{m}_{\sigma-2^n-1}$ -linked $\leq \operatorname{non}(\mathbf{SP}_n)$

Theorem

It is consistent with ZFC that $non(SP_n) < non(SP_{n+1})$.

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Theorem

There is a model of ZFC such that for every $n \in \omega$, non(**SP**_{*n*+1}) = $\mathfrak{m}_{2^n} = \omega_{n+1}$.

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What about the rest of the cardinal invariants associated?

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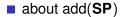
What about the rest of the cardinal invariants associated?

$$\mathsf{add}(\mathsf{SP}) = \mathit{min}\{|\mathcal{A}| : \mathcal{A} \subseteq \mathsf{SP} \land \bigcup \mathcal{A} \notin \mathsf{SP}\}$$

$$\mathsf{cov}(\mathsf{SP}) = \mathit{min}\{|\mathcal{A}| : \mathcal{A} \subseteq \mathsf{SP} \land \bigcup \mathcal{A} = \mathsf{2}^{\omega}\}$$

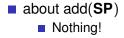
 $\mathsf{cof}(\mathsf{SP}) = \mathit{min}\{|\mathcal{A}| : \mathcal{A} \subseteq \mathsf{SP} \land \mathcal{A} \text{ is a cofinal family } \}$

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about add(SP)
 Nothing!
 about cof(SP)

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 Nothing!
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■ about cov(SP)

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about add(SP)

- Nothing!
- about cof(SP)
 - Nothing!
- about cov(SP)

It's easy to see that $cov(SP_3) \le cov(SP_2) \le cov(SP_1) = \mathfrak{c}$

about add(SP)

- Nothing!
- about cof(SP)
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It's easy to see that $cov(SP) \le \cdots \le cov(SP_3) \le cov(SP_2) \le cov(SP_1) = \mathfrak{c}$

Theorem

It is consistent with ZFC that $cov(SP_2) < c$

about add(SP)

- Nothing!
- about cof(SP)
 - Nothing!
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It's easy to see that $cov(SP_3) \leq cov(SP_2) \leq cov(SP_1) = \mathfrak{c}$

Theorem

It is consistent with ZFC that $cov(SP_2) < \mathfrak{c}$

Theorem (Hrušák, Zindulka)

It is consistent with ZFC that cof(N) < cov(SP).

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